MMP for 3-dimensional Kahler generalized pairs

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September 29, 2023

BGS

Recall:

Minimal Model Program: Let X be a projective variety with "good singularities". Then we want to find a birational maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n,$$

and a contraction $X_n \to Z$, with $\dim Z < \dim X_n$, and either $\dim Z > 0$ and the general fiber is Fano, or Z is a point and K_{X_n} is nef.

Cone theorem. Let X be a projective variety with klt singularities. Then there are countably many rational curves $C_j \subset X$ such that $0 < -K_X \cdot C_j \leq 2 \dim X$, and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

Contraction theorem. Let X be a projective variety with klt singularities. Let $F \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal face. Then there is a unique morphism $\mathrm{cont}_F \colon X \to Z$ to a projective variety such that $(\mathrm{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point if and only if $[C] \in F$.

Question. Can we find an equivalent statement for a Minimal Model Program if X is compact Kähler (but not projective)?

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ Y \end{tikzcd}$$

projective

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[Höring- Campana]

let X be a compact analytic variety. We need new definitions.

$$
\begin{array}{ccc}
N^1(X) & \longrightarrow & H^{1,1}_{\mathsf{BC}}(X) \\
N_1(X) & \longrightarrow & N_1(X) \\
\overline{\mathrm{NE}}(X) & \longrightarrow & \overline{\mathrm{NA}}(X) \\
\mathrm{Amp}(X) & \longrightarrow & \mathcal{K}(X)
\end{array}
$$

Replacing $N^1(X)$.

smooth function

The Bott – Chern cohomology $H^{1,1}_{\mathrm{BC}}(X)$ corresponds to the real d-closed $(1,1)$-forms with local potentials modulo $i\partial\bar{\partial}f$, or equivalently $H^{1,1}_{\mathrm{BC}}(X)$ corresponds to the d-closed bidegree $(1,1)$-currents with local potentials modulo $i\partial\bar{\partial}g$. For the analytic case, we say $N^1(X) := H^{1,1}_{\mathrm{BC}}(X)$. $\overset{\text{sm}}{\equiv} H^{1,1}(X)$

distribution

We can define $N_1(X)$ as the real d-closed currents of bi-dimension $(1,1)$ modulo the equivalence relation: $T_1 \equiv T_2$ if and only if $T_1(\eta) = T_2(\eta)$ for all real closed $(1,1)$ forms η with local potentials. $\overset{\text{sm}}{\equiv} H^{n-1,n-1}(X)$

We have that $N^1(X) = N_1(X)^*$.

Rat'l sing.

Replacing $\overline{\mathrm{NE}}(X)$. Kähler-Mori Cone

Define the Cone of currents $\overline{\mathrm{NA}}(X) \subseteq N_1(X)$ as the closeed cone generated by the positive closed currents of bi-dimension $(1,1)$.

We can see $\overline{\mathrm{NE}}(X) \subseteq \overline{\mathrm{NA}}(X)$ under the identification $C \mapsto T_C$, where $T_C(\eta) = \int_C \eta$.

Kähler varieties.

A Kähler variety is an analytic variety that carries a positive $(1,1)$ form ω such that locally on the smooth locus of X it can be written as $i\partial\bar{\partial}f$ for a plurisubharmonic smooth function f.

ω

We call such form a Kähler form.

Let $\mathcal{K}(X) \subset N^1(X)$ be the convex cone generated by the classes of Kähler forms. A class $\alpha \in N^1(X)$ is said to be nef if $\alpha \in \overline{\mathcal{K}}$.

$\Delta = 0$, terminal

Minimal model program for Kähler 3-folds. '20

Theorem. [Höring, Peternell '16], [Das, Hacon '23]
Let (X, Δ) be a $\mathbb{Q}$-factorial compact Kähler klt pair of dimension 3, such that $K_X + \Delta$ is pseudoeffective. Then there exists a minimal model program

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$$

such that $K_{X_n} + \Delta_n$ is nef.

Theorem. [Höring, Peternell '16], [Das, Hacon '23]
Let (X, Δ) be a $\mathbb{Q}$-factorial compact Kähler klt pair of dimension 3, such that $K_X + \Delta$ is pseudoeffective. Then there are at most countably many rational curves $\{\Gamma_i\}$ such that $-(K_X + \Delta) \cdot \Gamma_i \leq 6$ and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}^+[\Gamma_i].$$

Can we generalize the notion of pairs in order to get a more "Kähler" flavor?

b-$(1,1)$ currents.

Let X be a compact Kähler variety.

A closed b-$(1,1)$ current $\boldsymbol{\beta}$ is a collection of closed $(1,1)$ currents $\boldsymbol{\beta}_{X'}$ on all proper bimeromorphic models $X' \to X$, such that if $f\colon X_1 \to X_2$ is a bimeromorphic morphism of models of X, then $f_*\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$.

might

Notice that $\boldsymbol{\beta}_{X'}$ might not have local potentials. $[\beta_{X'}] \notin H^{1,1}_{BC}(X)$.

If β is a closed positive $(1,1)$ current with local potentials on X, then we can define a b-$(1,1)$ current $\bar{\beta}$, by assigning to each model $\nu\colon X' \to X$ the $(1,1)$ current $\bar{\beta}_{X'} := \nu^*\beta$.

If $\boldsymbol{\beta} = \bar{\beta}$ for some $(1,1)$ current β on X, then we say that $\boldsymbol{\beta}$ descends to X.

Generalized pairs for Kähler varieties.

Let X be a compact Kähler variety.

Let $\nu\colon X' \to X$ a resolution, B' and $\mathbb{R}$-divisor on X' with SNC support such that $B := \nu_* B' \geq 0$, and $\boldsymbol{\beta}$ a closed b-$(1,1)$ current. We say that $(X, B + \boldsymbol{\beta})$ is a generalized pair if

- $\boldsymbol{\beta}$ is a positive closed b-$(1,1)$ current that descends to X',
- $[\boldsymbol{\beta}_{X'}] \in H^{1,1}_{\mathsf{BC}}(X')$ is nef,
- $[K_{X'} + B' + \boldsymbol{\beta}_{X'}] = \nu^*\gamma$, for some $\gamma \in H^{1,1}_{\mathsf{BC}}(X)$.

Given $(X, B + \boldsymbol{\beta})$ and $\beta = \boldsymbol{\beta}_{X'}$, then B' is uniquely determined.

A similar definition can be given for the relative setup.

Singularities.

Let P be a Weil divisor over X. Define the generalized discrepancy $a(P, X, B + \boldsymbol{\beta})$ as follows. Let $\nu \colon X' \to X$ be a log resolution of $(X, B + \boldsymbol{\beta})$ such that $P \subseteq X'$. Then $a(P, X, B + \boldsymbol{\beta}) = -\mathrm{mult}_P(B')$.

gklt

We say that $(X, B + \boldsymbol{\beta})$ is generalized klt if $a(P, X, B + \boldsymbol{\beta}) > -1$.

We say that $(X, B + \boldsymbol{\beta})$ is generalized lc if $a(P, X, B + \boldsymbol{\beta}) \geq -1$.

We say that $(X, B + \boldsymbol{\beta})$ is generalized dlt if there is an open set $U \subseteq X$ such that $(U, (B + \boldsymbol{\beta})|_U)$ is a log resolution, $-1 \leq a(P, X, B + \boldsymbol{\beta}) \leq 0$ for any prime divisor P on U, and $-1 < a(P, X, B + \boldsymbol{\beta})$ for any prime divisor P over X with center in $X \setminus U$.

If $(X, B + \boldsymbol{\beta})$ is a gklt pair, then X has rational singularities.

β_X is big

Theorem. [Das, Hacon, Y. '23]
Let $(X, B + \boldsymbol{\beta})$ be a gklt pair, where X is a compact Kähler 3-fold. Assume that $K_X + B + \boldsymbol{\beta}$ is big. Then $(X, B + \boldsymbol{\beta})$ has a log canonical model, and there exist a log terminal model, and all such models admit a morphism to the log canonical model.

Proof relies heavily on the MMP for (X, Δ)

Sketch of the proof.

- First, reduce to the case $\boldsymbol{\beta}_X$ Kähler and (X, B) log smooth.

- We have that $K_X + B + \boldsymbol{\beta}_X$ is pseudoeffective and that $K_X + B + (1+t)\boldsymbol{\beta}_X$ is Kähler for $t \gg 0$. Under this setup, we can run a $K_X + B + \boldsymbol{\beta}_X$-MMP with scaling of $t\boldsymbol{\beta}_X$ that terminates in a log terminal model.

Modified Kähler

- Let $X \dashrightarrow X^m$ be such model, with $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$.

- Again, borrowing from the MMP for pairs, we can contract and flip all the $K_{X^m} + B^m + \boldsymbol{\beta}_{X^m}$-trivial curves that are $K_{X^m} + B^m$-negative. We obtain $X^m \dashrightarrow X^n$.

[Collins, tosatti]

- From [Das, Hacon '20], $\mathrm{Null}(K_{X^n} + B^n + \boldsymbol{\beta}_{X^n})$ is a union of curves, and they can be contracted.

- Let $X^n \to Z$ be the morphism obtained from contracting $\mathrm{Null}(K_{X^n} + B^n + \boldsymbol{\beta}_{X^n})$, then Z is the log canonical model, and the map $X^m \to Z$ is also a morphism.

More results.

- Cone theorem for $\overline{\mathrm{NA}}(X)$ in terms of $K_X + B + \boldsymbol{\beta}_X$.

- If $K_X + B + \boldsymbol{\beta}_X$ is not big, then we obtain a contraction after running the MMP. If $K_X + B + \boldsymbol{\beta}_X$ is not pseudoeffective then we obtain a Mori fiber space.

- Finiteness of some minimal models, and local polyhedral decomposition of space of closed positive $(1, 1)$ currents (analogue to results from [BCHM])

- Minimal models are connected by flips, anti flips and flops.

Thank you!